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On a periodic soliton cellular automaton

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Abstract
We propose a box and ball system with a periodic boundary condition periodic box and ball system (pBBS). The time evolution rule of the pBBS is represented as a Boolean recurrence formula, an inverse ultradiscretization of which is shown to be equivalent to the algorithm of the calculus for the $2N$th root. The relations to the pBBS of the combinatorial R matrix of $U'_q(A_N^{(1)})$ are also discussed.

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1. Preface

In many physical phenomena, physical quantities and time–space variables are continuous. Discretized models, however, are used sometimes for simplification and/or speedup of computation. In the discretizing process, it is preferable to keep the mathematical structure (symmetry, conserved quantities, etc) of the original continuous systems. For integrable systems, i.e. those systems which exhibit solitonic natures and are described by integrable nonlinear partial differential equations, several *effective* methods of discretization have been established and many discrete integrable systems have been proposed [1]. An effective method to create a cellular automaton (CA) with solitonic nature is the ultradiscretization (UD) [2], which is a limiting procedure to discretize the dependent variables of a partial difference equation.

In this paper, we consider an integrable CA called the box and ball system (BBS) [3, 4]. The BBS has been defined as the time evolution of a finite number of balls moving in a one-dimensional array of an *infinite* number of boxes. We extend the time evolution rule of the BBS so that the BBS can be defined when we impose a periodic boundary condition. We show that the time evolution rule of the periodic BBS (pBBS) is given by a Boolean formula, which turns out to be an algorithm to calculate the $2N$th root of a given number through inverse UD. It is well known that many of the discrete integrable systems are equivalent to good algorithms such as the diagonalization of a matrix (QR algorithm, etc), convergence acceleration algorithm

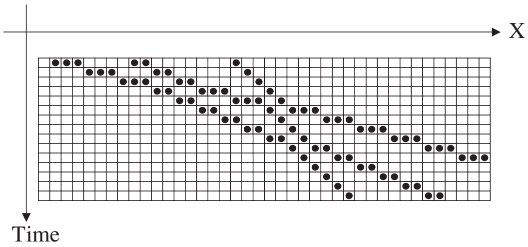


Figure 1. An example of the time evolution of BBS.

(ε algorithm, etc) [6], Karmarkar algorithm or BCH-Goppa decoding. Our Boolean formula is an example of such a correspondence between discrete integrable systems and algorithms.

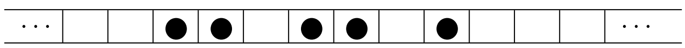
In section 2, we introduce the pBBS and explain its time evolution rule. In section 3, we obtain recurrence formulae on Boolean algebras for the time evolution rule of the simplest BBS with a box capacity of one. The correspondence of the Boolean formula with the algorithm for the computation of the $2N$th root is shown. In section 4, we establish a relation of the pBBS to the combinatorial R-matrix model and extend the time evolution rule to the general pBBSs with various box capacities. Section 5 is devoted to some concluding remarks.

2. Box and ball system

2.1. Infinite BBS

The BBS is a reinterpretation and extension of the filter-type CA proposed by Takahashi and Satsuma [2, 5].

Let us consider a one-dimensional array of an infinite number of boxes. At the initial time $t = 0$, all but a finite number of boxes are empty, and each of the rest boxes contains one ball:



The time evolution rule of this system from time t to $t + 1$ is given as follows:

(1) Move every ball only once.
(2) Move the leftmost ball to its nearest right empty box.
(3) Move the leftmost ball of the rest ball to its nearest right empty box.
(4) Repeat the above procedure until all the balls are moved.

An example of the time evolution is shown in figure 1. In this example, we see the solitonic behaviour of the balls. This is a general behaviour of the BBS and, starting from an arbitrary initial state, we can prove that the state asymptotically evolves into the state that consists of only freely moving solitons. It is also known that the BBS has an infinite number of conserved quantities [8]. By replacing an empty box by ‘0’ and a filled box by ‘1’, we regard the BBS as a dynamical system of ‘01’ sequences, namely a CA. The relation between BBS and (classical) integrable systems, i.e. integrable partial differential and/or difference equations, was established by the notion of UD [2]. In the approach of UD, the BBS is constructed through a limit of the partial difference equation, which is obtained by reduction of the discrete KP equation (Hirota–Miwa equation). Since the discrete KP equation is equivalent to the generating formula of the KP hierarchy, the BBS naturally inherits its integrability and shows a solitonic nature [7]. In this sense, the BBS is called an integrable CA.

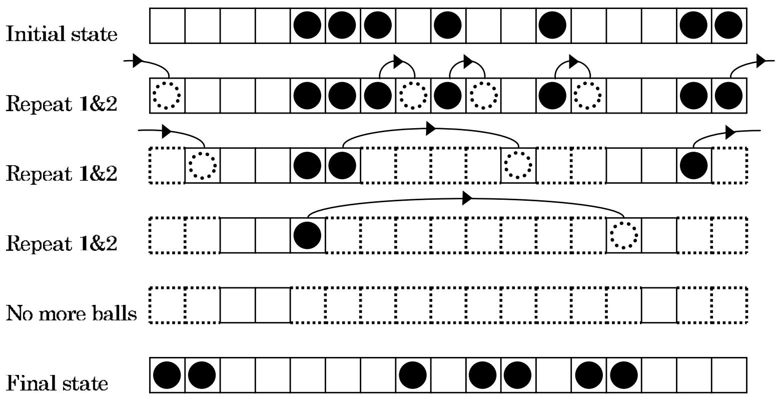


Figure 2. An example of the modified rule for pBBS.

2.2. Periodic BBS (pBBS)

In this subsection, we extend the original BBS to that with a periodic boundary condition. Let the BBS consist of N boxes. To impose a periodic boundary condition, we assume that the Nth box is the adjacent box to the first box. (We may imagine that the boxes are arranged in a circle.) We also assume that the box capacity is 1 for all the boxes. Since the evolution rule of the BBS requires the definition of the *leftmost* ball in the BBS, we cannot apply the original evolution rule directly to a periodic system. Instead, we consider the following time evolution rule:

(1) Move all the balls to their adjacent right boxes if the boxes are empty.
(2) Ignore the boxes to which and from which the balls were moved in the first step, and move all of the remaining balls to their adjacent right boxes if they are empty.
(3) Repeat the above procedure until all the balls are moved.

An example of the movement of the balls by this rule is illustrated in figure 2.

We see that this rule is equivalent to the original time evolution rule when we apply it to the BBS with an infinite array of boxes. In this rule, the number of balls which move at each stage does not change in time evolution. We will denote by a pBBS the BBS with a periodic boundary condition which evolves by this rule.

Example 1 (pBBS ($N = 7$)). Time evolution patterns of pBBS are shown in figure 3. (a) The initial state is '1110000'. The time evolution pattern has the fundamental cycle 7. (b) The initial state '1101000' has the fundamental cycle 21.

The time evolution of pBBS is represented by a diagram with N nodes and M lines, each of which connects two of the distinct nodes as shown in figure 4.

Each node corresponds to a box of the pBBS and a line is drawn from the box which contains a ball to the box to which the ball is moved. From the time evolution rule of the pBBS, we find the following properties of the diagram:

- There is no intersection of the lines.
- There is no line over the nodes which are not connected by lines.
- If we remove the unconnected nodes, the diagram is divided into a set of *blocks* with an even number of nodes which are connected in pairs by lines.

For convenience of explanation in the subsequent sections, we define indices $\{\gamma_t(n)\}_{n=1}^{N}$ of the nodes in the diagram at time step t:

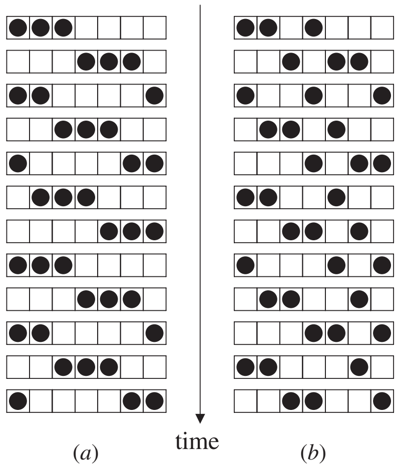


Figure 3. Examples of pBBS ($N = 7$).

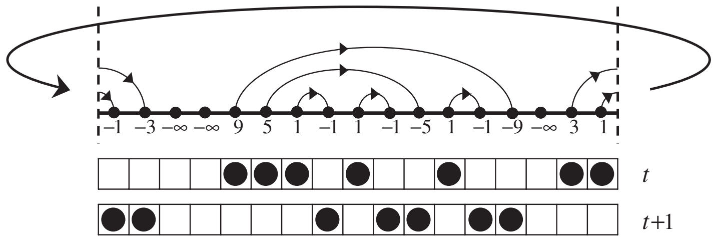


Figure 4. A diagram of time evolution and the corresponding time evolution of the pBBS. The integers below the nodes are the indices $\gamma_t(n)$.

- If a ball in the nth box is moved to the $(n+k)$th (modulo N) box at $t+1$, $\gamma_t(n) = k(\in \mathbb{Z}_{>0})$.
- If a ball is moved to the nth box from the $(n-k)$th (modulo N) box at $t+1$, $\gamma_t(n) = -k$.
- Otherwise, $\gamma_t(n) = -\infty$.

In the example shown in figure 4, the indices $\{\gamma_t(n)\} = \{ -1, -3, -\infty, -\infty, 9, 5, 1, -1, 1, -1, -5, 1, -1, -9, -\infty, 3, 1 \}$. The following proposition is a direct consequence of the properties of the diagram.

Proposition 1. *Let k and l ($l < k$) be positive integers. Then*

(a) If $\gamma_t(n) = k$, then $\gamma_t(n+k) = -k$, and $-(k-2) \leqslant \gamma_t(n+j) \leqslant k-2$ for $1 \leqslant j \leqslant k-1$.
(b) Furthermore if there exists j such that $1 \leqslant j \leqslant k-1$ and $\gamma_t(n+j) = -l$, then $l < j$ and $\gamma_t(n+j-l) = l$.

In proposition 1, the indices are understood with the convention of *modulo N*, i.e. $\gamma_t(n) \equiv \gamma_t(n+N)$. Hereafter, we often use this convention.

There are several equivalent evolution rules for the pBBS. For example,

(1) At each filled box, create a copy of the ball.
(2) Move all the copies once according to the following rule.
(3) Choose one of the copies and move it to its nearest right empty box.
(4) Choose one of the rest copies and move it to its nearest right empty box.
(5) Repeat the above procedure until all the copies are moved.
(6) Delete all the original balls.

It is not difficult to prove that the result does not depend on the choice of the copies at each stage, and that this rule gives the same time evolution pattern of the previous rule. An advantage of this rule is that it is straightforward to extend the rule to the BBS with many kinds of balls and various box capacities. We can also use the combinatorial R-matrix of $U'_q(A^{(1)}_{N-1})$ to give an equivalent evolution rule, the details of which are presented in section 4.

3. Recurrence equations and corresponding algorithm

3.1. Boolean formulae of pBBS

We show that the pBBS introduced above can be formulated by Boolean algebra. Let N be the number of boxes. The space of the states of pBBS is naturally regarded as ${\mathbb{F}_2}^N$. We denote a state $X(t)$ of the pBBS at time t by $X(t) = (x_1(t), x_2(t), \ldots, x_N(t)) \in {\mathbb{F}_2}^N$, where $x_i(t) = 0$ if the ith box is empty and $x_i(t) = 1$ if it is filled. Let $\wedge$, $\vee$ and $\oplus$ be AND, OR and XOR respectively. These Boolean operators are realized in ${\mathbb{F}_2}^N$ as the map: ${\mathbb{F}_2}^N \times {\mathbb{F}_2}^N \to {\mathbb{F}_2}^N$. For $X = (x_1, x_2, \ldots, x_N)$, $Y = (y_1, y_2, \ldots, y_N)$, they are defined as

$$(X \wedge Y)_i := x_i \wedge y_i \equiv x_i y_i$$
$$(X \vee Y)_i := x_i \vee y_i \equiv x_i y_i + x_i + y_i$$
$$(X \oplus Y)_i := x_i \oplus y_i \equiv x_i + y_i.$$

We also define rotate shift to the right S:

$$SX = (x_N, x_1, x_2, \ldots, x_{N-1}).$$

The next theorem gives an expression of T:

$$T : X(t) \mapsto X(t+1)$$

in terms of these Boolean operators.

Theorem 1. *Suppose that $X(t) \in {\mathbb{F}_2}^N$ is the state of the pBBS at time step t. We consider the following recurrence equations:*

$$A^{(0)} = X(t), \qquad B^{(0)} = SX(t), \tag{1}$$

$$\begin{aligned} A^{(n+1)} &:= A^{(n)} \vee B^{(n)} \\ B^{(n+1)} &:= S(A^{(n)} \wedge B^{(n)}) \end{aligned} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

Then

$$X(t+1) = A^{(N)} \oplus X(t), \qquad \text{and} \qquad B^{(N)} = \mathbf{0}, \tag{3}$$

where $\mathbf{0} := (0, 0, \ldots, 0)$.

Proof 1. We define $D^{(i)} = (d_1^{(i)}, d_2^{(i)}, \ldots, d_N^{(i)}) \in {\mathbb{F}_2}^N$ $(i = 1, 2, \ldots, N)$ as follows. If the lth box contains a ball at time step t and it moves to the $(l+i)$th box at $t+1$, then $d_l^{(i)} = 1$, otherwise $d_l^{(i)} = 0$. If we use the indices $\gamma_t(l)$ defined in the previous section, the definition of $D^{(i)}$ is rewritten as

$$d_l^{(i)} = 1 \qquad \text{if and only if } \gamma_t(l) = i.$$

Clearly, these $D^{(i)}$ give the unique decomposition of the initial state $X(t)$ and the final state $X(t+1)$ as

$$X(t) = \bigoplus_{i=1}^{N} D^{(i)}, \qquad D^{(i)} \wedge D^{(j)} = \mathbf{0}\ (i \neq j),$$

$$TX(t) = \bigoplus_{i=1}^{N} S^i D^{(i)}, \qquad S^i D^{(i)} \wedge S^j D^{(j)} = \mathbf{0}\ (i \neq j),$$

where $\bigoplus_{i=1}^{N} D^{(i)} := D^{(1)} \oplus D^{(2)} \oplus \cdots \oplus D^{(N)}$.

To prove the theorem, it suffices to prove the following formulae:

$$A^{(n)} = X(t) \oplus \bigoplus_{i=1}^{n} S^i D^{(i)}, \tag{4}$$

$$S^{-n-1} B^{(n)} = X(t) \oplus \bigoplus_{i=1}^{n} D^{(i)}. \tag{5}$$

Indeed, if (4) and (5) hold, we have

$$\begin{aligned} A^{(N)} \oplus X(t) &= X(t) \oplus \bigoplus_{i=1}^{N} S^i D^{(i)} \oplus X(t) \\ &= \bigoplus_{i=1}^{N} S^i D^{(i)} \\ &= X(t+1), \\ B^{(N)} &= S^{N+1}\left(X(t) \oplus \bigoplus_{i=1}^{N} D^{(i)}\right) \\ &= S^{N+1}(X(t) \oplus X(t)) \\ &= \mathbf{0}. \end{aligned}$$

We prove (4) and (5) by induction.

For $n = 1$,

$$\begin{aligned} A^{(1)} &= A^{(0)} \vee B^{(0)} \\ &= X(t) \vee SX(t) \\ &= X(t) \vee SD^{(1)} \vee \bigoplus_{i=2}^{N} SD^{(i)}. \end{aligned}$$

Suppose that $\bigoplus_{i=2}^{N} SD^{(i)} \vee X(t) \neq X(t)$, there exists $D^{(q)} (q \geqslant 2)$ such that its n' component $d_{n'}^{(q)} = 1$ and $\gamma_t(n'+1) \leqslant -1$. However, by the definition of $D^{(q)}$, $d_{n'}^{(q)} = 1$ implies $\gamma_t(n') = q (\geqslant 2)$. From proposition 1(a), $-q+2 \leqslant \gamma_t(n'+1) \leqslant q-2$ and we find $-q+2 \leqslant \gamma_t(n'+1) \leqslant -1$, which contradicts proposition 1(b). Hence $\bigoplus_{i=2}^{N} SD^{(i)} \vee X(t) = X(t)$. Since $SD^{(1)} \wedge X(t) = \mathbf{0}$ follows from $TX(t) \wedge X(t) = \mathbf{0}$, we find

$$A^{(1)} = X(t) \oplus SD^{(1)}.$$

Similarly, from the relations $X(t) \wedge \bigoplus_{i=2}^{N} SD^{(i)} = \bigoplus_{i=2}^{N} SD^{(i)}$ and $SD^{(1)} \wedge X(t) = \mathbf{0}$, we have

$$\begin{aligned} S^{-2} B^{(1)} &= S^{-1}(X(t) \wedge SX(t)) \\ &= S^{-1}\left(X(t) \wedge \bigoplus_{i=1}^{N} SD^{(i)}\right) \\ &= S^{-1}(X(t) \wedge SD^{(1)}) \oplus S^{-1}\left(X(t) \wedge \bigoplus_{i=2}^{N} SD^{(i)}\right) \\ &= (S^{-1}\mathbf{0}) \oplus S^{-1}\left(\bigoplus_{i=2}^{N} SD^{(i)}\right) \\ &= \bigoplus_{i=2}^{N} D^{(i)} \\ &= X(t) \oplus D^{(1)}. \end{aligned}$$

Hence (4) and (5) hold for $n = 1$.

Assume that (4) and (5) is true for $n = k$, then

$$A^{(k+1)} = \left(X(t) \oplus \bigoplus_{i=1}^{k} S^i D^{(i)}\right) \vee \left(S^{k+1} X(t) \oplus \bigoplus_{i=1}^{k} S^{k+1} D^{(i)}\right)$$
$$= \left(X(t) \oplus \bigoplus_{i=1}^{k} S^i D^{(i)}\right) \vee \left(\bigoplus_{i=k+1}^{N} S^{k+1} D^{(i)}\right).$$

By the definition of $D^{(k+1)}$, we have

$$\left(X(t) \oplus \bigoplus_{i=1}^{k} S^i D^{(i)}\right) \vee S^{k+1} D^{(k+1)} = X(t) \oplus \bigoplus_{i=1}^{k+1} S^i D^{(i)}.$$

Suppose that

$$\left(X(t) \oplus \bigoplus_{i=1}^{k} S^i D^{(i)}\right) \vee \bigoplus_{i=k+2}^{N} S^{k+1} D^{(i)} \neq X(t) \oplus \bigoplus_{i=1}^{k} S^i D^{(i)}.$$

Noticing the fact that, for $X(t) \oplus \bigoplus_{i=1}^{k} S^i D^{(i)} = (a_1^{(k)}, a_2^{(k)}, \ldots, a_N^{(k)})$, $a_j^{(k)} = 1$ if and only if $-k \leqslant \gamma_t(j)$, we find that there is at least one $D^{(q)}$ $(q \geqslant k+2)$, one of whose components satisfies $d_{n'}^{(j)} = 1$ and $\gamma_t(n'+k+1) \leqslant -k-1$. However, by the definition of $D^{(q)}$, $d_{n'}^{(q)} = 1$ implies $\gamma_t(n') = q(\geqslant k+2)$. From proposition 1(a), $-q+2 \leqslant \gamma_t(n'+k+1) \leqslant q-2$ and we find $-q+2 \leqslant \gamma_t(n'+k+1) \leqslant -k-1$, which contradicts proposition 1(b). Thus, we have

$$A^{(k+1)} = X(t) \oplus \bigoplus_{i=1}^{k+1} S^i D^{(i)}.$$

In a similar manner, we also obtain

$$S^{-k-2} B^{(k+1)} = X(t) \oplus \bigoplus_{i=1}^{k+1} D^{(i)}.$$

Hence, from the assumption of induction, the formulae (4) and (5) are proved to hold for $0 \leqslant n \leqslant N-1$, which completes the proof of the theorem. □

This recurrence equation (3) is expressed with only three operations, AND, OR and SHIFT, and has a simple form. The SHIFT operator introduces the right-and-left symmetry breaking that comes from the definition of the direction of the movement of the balls.

Since $D^{(2m)} = \mathbf{0}$, we immediately obtain the following corollary:

Corollary 1. *Suppose that $X(t) \in \mathbb{F}_2{}^N$ is given as the state at time t. Then the state at the next time $X(t+1) = TX(t)$ is calculated by the recurrence equation as follows.*

$$A^{(0)} := X(t), \qquad B^{(0)} := SX(t) \tag{6}$$
$$A^{(n+1)} := A^{(n)} \vee B^{(n)} \qquad B^{(n+1)} := S^2(A^{(n)} \wedge B^{(n)}), \tag{7}$$

and

$$X(t+1) = A^{(\lfloor N/2 \rfloor)} \oplus X(t), \qquad B^{(\lfloor N/2 \rfloor)} = \mathbf{0}. \tag{8}$$

3.2. pBBS and numerical algorithm

The formulae for time evolution of the pBBS (2) have a simple and symmetric form, and we expect that they have some relationship to a good algorithm. In this subsection, we show that they have indeed the same structure as that of the algorithm to compute the Nth root of a given number. Henceforth, let the truth value '0 (false)' and '1 (true)' be equivalent to the integer $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$. Then we can replace $\wedge$ and $\vee$ with min and max as

$$x \wedge y \iff \min[x, y]$$
$$x \vee y \iff \max[x, y].$$

Following the notation of the previous section, we define that max and min act on $\mathbb{Z}^N$ bitwise. Then, equation (2) can be rewritten by the integer equation as

$$\begin{aligned} A^{(n+1)} &= \max[A^{(n)}, B^{(n)}] \\ B^{(n+1)} &= S\min[A^{(n)}, B^{(n)}]. \end{aligned} \tag{9}$$

We construct the difference equations corresponding to (9) by means of inverse UD [2]. Noticing the identity

$$\max[x, y] = \lim_{\epsilon\to+0} \epsilon \log(e^{x/\epsilon} + e^{y/\epsilon}) \qquad (x, y \in \mathbb{R}),$$

and $\min[x, y] = -\max[-x, -y]$, we think of the difference equations:

$$\begin{aligned} a_i^{(n+1)} &= \{a_i^{(n)} + b_i^{(n)}\}/2 \\ b_i^{(n+1)} &= 2\{(a_{i-1}^{(n)})^{-1} + (b_{i-1}^{(n)})^{-1}\}^{-1} \end{aligned} \qquad (1 \leqslant i \leqslant N), \tag{10}$$

where the subindex i is taken modulo N. The relation between (9) and (10) is obvious. When we replace $a_i^{(n)}$ and $b_i^{(n)}$ with $e^{(A^{(n)})_i/\epsilon}$ and $e^{(B^{(n)})_i/\epsilon}$, respectively, and take a limit $\epsilon \to +0$, we obtain (9) from (10). The factor 2 in (10) is so chosen that the recurrence formulae do not diverge at $n \to \infty$.

When we disregard the space coordinates i in (10), or consider the case $N = 1$, we have the recurrence formula

$$\begin{aligned} a^{(n+1)} &= \frac{a^{(n)} + b^{(n)}}{2} \\ b^{(n+1)} &= \frac{2a^{(n)}b^{(n)}}{a^{(n)} + b^{(n)}}, \end{aligned} \tag{11}$$

which is the well-known arithmetic-harmonic mean algorithm [11] and we have

$$\lim_{n\to\infty} a^{(n)} = \lim_{n\to\infty} b^{(n)} = \sqrt{a^{(0)}b^{(0)}}.$$

The recurrence formula (10) for general N is also considered as a numerical algorithm to calculate the $2N$th root of a given number. To see this, first we note that (10) has a conserved quantity C with respect to the step n:

$$C^{(n)} := \prod_{i=1}^{N} a_i^{(n)} b_i^{(n)} = C^{(n-1)} = \cdots = C^{(0)} = \prod_{i=1}^{N} a_i^{(0)} b_i^{(0)} \equiv C, \tag{12}$$

where $\{a_i^{(0)}, b_i^{(0)}\}$ are the initial values. Then we can show the following proposition:

Proposition 2. *If all the initial values $\{a_i^{(0)}, b_i^{(0)}\}$ are positive, then they converge to the same value*

$$\lim_{n\to\infty} a_k^{(n)} = \lim_{n\to\infty} b_k^{(n)} = \sqrt[2N]{\prod_{i=1}^{N} a_i^{(0)} b_i^{(0)}} = \sqrt[2N]{C} \qquad \textit{(for all } k\textit{)}.$$

Hence, the recurrence formula of pBBS is regarded as a numerical algorithm of the $2N$th root.

To prove the proposition, we need a lemma.

Lemma 1. *For $m > 0$, $\alpha \geqslant 0$ and $\varepsilon > 0$ which satisfy*

$$(2\alpha^2 + 5\alpha + 4)\varepsilon < m, \tag{13}$$

if it holds that $m - \varepsilon < a_i^{(n+1)} < m + \alpha\varepsilon$, $m - \varepsilon < a_i^{(n)}$ and $m - \varepsilon < b_i^{(n)}$, then $a_i^{(n)}, b_i^{(n)} < m + (2\alpha + 2)\varepsilon$. Similarly, if it holds that $m - \varepsilon < b_{i+1}^{(n+1)} < m + \alpha\varepsilon$, $m - \varepsilon < a_i^{(n)}$ and $m - \varepsilon < b_i^{(n)}$, then $a_i^{(n)}, b_i^{(n)} < m + (2\alpha + 2)\varepsilon$.

This lemma is proved from (10) by straightforward calculations. Now we give the proof of the proposition 2.

Proof 2. Let $m^{(n)} := \min_{i=1,\dots,N}[a_i^{(n)}, b_i^{(n)}]$ and $M^{(n)} := \max_{i=1,\dots,N}[a_i^{(n)}, b_i^{(n)}]$. Since we have from (10)

$$\min[a_i^{(n)}, b_i^{(n)}] \leqslant a_i^{(n+1)} \leqslant \max[a_i^{(n)}, b_i^{(n)}]$$
$$\min[a_i^{(n)}, b_i^{(n)}] \leqslant b_{i+1}^{(n+1)} \leqslant \max[a_i^{(n)}, b_i^{(n)}],$$

we obtain

$$m^{(n)} \leqslant m^{(n+1)} \leqslant M^{(n+1)} \leqslant M^{(n)} \qquad \text{for } n = 0, 1, 2, \dots. \tag{14}$$

From (14), we find $^{\exists}m,\ m = \lim_{n\to\infty} m^{(n)}$ and $^{\exists}M,\ M = \lim_{n\to\infty} M^{(n)}$. Clearly $m \geqslant m^{(n)}$, $M \leqslant M^{(n)}$ and $m \leqslant M$.

We will prove $m = M$ by finding a contradiction to the assumption $m < M$. Since $m^{(n)}$ and $M^{(n)}$ converge to m and M, respectively, for all $\varepsilon > 0$, there exists n_0 such that $0 \leqslant m - m^{(n)} < \varepsilon$ and $0 \leqslant M^{(n)} - M < \varepsilon$ for $^{\forall}n \geqslant n_0$. We take $\varepsilon = \min[(M - m)/2^{N+1},\ m/2^{2N+4}]$. Note that, with this choice, the inequality (13) holds for $0 \leqslant \alpha \leqslant 2^{N+1}$. For $n = n_0 + N$, there exists $a_j^{(n_0+N)}$ or $b_j^{(n_0+N)}$ which is equal to m^{n_0+N}. When $a_j^{(n_0+N)} = m^{(n_0+N)}$, noticing $m - \varepsilon \leqslant m^{(n_0+N)} \leqslant m$, $m - \varepsilon \leqslant a_j^{(n_0+N-1)}$ and $m - \varepsilon \leqslant b_j^{(n_0+N-1)}$, we obtain from the lemma 1 that $m - \varepsilon \leqslant a_j^{(n_0+N-1)}, b_j^{(n_0+N-1)} \leqslant m + 2\varepsilon$. Similarly, when $b_j^{(n_0+N)} = m^{(n_0+N)}$, we obtain $m - \varepsilon \leqslant a_{j-1}^{(n_0+N-1)}, b_{j-1}^{(n_0+N-1)} \leqslant m + 2\varepsilon$. By repeated use of the lemma 1 in a similar manner, we finally obtain

$$^{\forall}i\ a_{\mathrm{i}}^{(n_0)} \leqslant m + (2^{N+2} - 2)\varepsilon < M \leqslant M^{(n_0)}$$
$$^{\forall}i\ b_{\mathrm{i}}^{(n_0)} \leqslant m + (2^{N+2} - 2)\varepsilon < M \leqslant M^{(n_0)},$$

which contradicts the definition of $M^{(n_0)}$. Thus, we have proved $m = M$. Hence all the values converge to the same value $C^{1/2N}$. □

In the preface to this paper, we pointed out that the discrete model has to maintain the mathematical structures of a continuous model in the process of UD. When we take the ultradiscrete limit of C, it is also a conserved quantity of the pBBS. In fact,

$$C = \prod_{i=1}^{N} a_i^{(0)} b_i^{(0)} \overset{\text{UD}}{\Longrightarrow} \sum_{i=1}^{N} \{A_i^{(0)} + B_i^{(0)}\} \tag{15}$$

gives the double number of balls in the pBBS. The number of balls is, to be sure, a conserved quantity of the pBBS.

We can construct other conserved quantities of the recurrence formulae (2) by means of another inverse UD.

From (2), we have

$$\begin{aligned} A^{(n+1)} \wedge S^{-1} B^{(n+1)} &= A^{(n)} \wedge B^{(n)} \\ A^{(n+1)} \vee S^{-1} B^{(n+1)} &= A^{(n)} \vee B^{(n)}. \end{aligned} \tag{16}$$

When we consider the inverse UD of the above equation, we have

$$
\begin{aligned}
&a_i^{(n+1)} b_{i+1}^{(n+1)} = a_i^{(n)} b_i^{(n)} \\
&a_i^{(n+1)} + b_{i+1}^{(n+1)} = a_i^{(n)} + b_i^{(n)}.
\end{aligned}
\tag{17}
$$

Thus, for arbitrary λ, $(\lambda + a_i^{(n)})(\lambda + b_i^{(n)}) = (\lambda + a_i^{(n+1)})(\lambda + b_{i+1}^{(n+1)})$ and we find that

$$
C_n(\lambda) := \prod_{i=1}^{N} (\lambda + a_i^{(n)})(\lambda + b_i^{(n)}) \tag{18}
$$

does not depend on n, which means that any symmetric polynomial with respect to $\{a_i^{(n)}\}$ and $\{b_i^{(n)}\}$ does not depend on n. Therefore, the ultradiscrete limit of such symmetric polynomials gives $2N$ conserved quantities $S_1, S_2, \ldots, S_{2N}$ of (2). If we denote $B_i^{(n)} \equiv A_{N+i}^{(n)}$, these conserved quantities are explicitly given as

$$
\begin{aligned}
&S_1 := \max_i [A_i^{(n)}] \\
&S_2 := \max_{i<j} [A_i^{(n)} + A_j^{(n)}] \\
&\cdots \\
&S_{2N} := \sum_{i=1}^{2N} A_i^{(n)}.
\end{aligned}
$$

4. pBBS and combinatorial R matrix

4.1. pBBS as a periodic $A_M^{(1)}$ crystal lattice

The BBS (of an infinite number of boxes) has recently been reformulated from the crystal theory and the combinatorial R matrix [9, 12, 13]. In this approach, a time evolution pattern of the BBS corresponds to a ground state configuration of a solvable lattice which has a symmetry of quantum algebra $U'_q(A_M^{(1)})|_{q\to 0}$. The Boltzmann weight on every vertex of the lattice is given by a combinatorial R-matrix of $U'_q(A_n^{(1)})$, and the states on each link are represented as the M-fold symmetric tensor representation B_M. For the simplest BBS, in which only one kind of ball exists and all the box capacities are 1, the lattice model has the space of horizontal links $B_1{}^{\otimes\infty} \equiv \cdots \otimes B_1 \otimes B_1 \otimes \cdots$, and that of the vertical links $B_\infty{}^{\otimes\infty} \equiv \cdots \otimes B_\infty \otimes B_\infty \otimes \cdots$. Here B_∞ is understood as $B_\infty = B_N$ ($N \gg 1$). Precisely speaking, N can be any positive integer which is greater than the number of the balls in the BBS. The combinatorial R-matrix of $U'_q(A_1^{(1)})$ gives an isomorphism $B_\infty \otimes B_1 \to B_1 \otimes B_\infty$. The initial condition of the BBS corresponds to the initial state of the horizontal links of the lattice model. Since the number of balls is finite, the initial state $\cdots \otimes |u_{n-1}^{t=0}\rangle \otimes |u_n^{t=0}\rangle \otimes |u_{n+1}^{t=0}\rangle \otimes \cdots$ ($\in B_1{}^{\otimes\infty} \equiv \cdots \otimes B_1 \otimes B_1 \otimes \cdots$) satisfies the condition

$$
|u_n^{t=0}\rangle = |0\rangle \qquad \text{for } |n| \gg 1,
$$

where $|0\rangle$ denotes the highest weight vector of B_1. (The basis of B_1 will be denoted by $|0\rangle$ and $|1\rangle$, where $|0\rangle$ corresponds to an empty box and $|1\rangle$ corresponds to a box with a ball.) The boundary condition for the vertical links is expressed as $|v_n^t\rangle = |\{0\}\rangle$ for ($|n| \gg 1$), where $|\{0\}\rangle$ is the highest weight vector of B_∞. In general, we can replace B_1 with B_{θ_n} and B_∞ with B_{κ_t}, where θ_n is the box capacity of the nth box, and κ_t is the carrier capacity of the tth carrying cart [10, 13]. The solitonic natures of the BBS can be proved algebraically with the above setting [14] and the BBS can be extended to other quantum algebras [15].

The pBBS discussed in the previous sections is also reformulated as a combinatorial R-matrix lattice model with periodic boundary condition. For the original BBS, time evolution is given by the isomorphism:

$$\mathcal{T}: \ B_\infty \otimes B_1^{\otimes N} \to B_1^{\otimes N} \otimes B_\infty$$
$$\mathcal{T}: \ |\{0\}\rangle \otimes |c(t)\rangle \mapsto |c(t+1)\rangle \otimes |\{0\}\rangle$$

where $|c(t)\rangle \in B_1^{\otimes N}$ is the state corresponding to the BBS at time t. For the pBBS, we have to take the trace of the vertical state, i.e. by regarding $\mathcal{T} \in \mathrm{End}_{\mathrm{End} B_1^{\otimes N}} B_\infty$, we define the matrix $T := \mathrm{Tr}_{B_\infty} \mathcal{T} \in \mathrm{End}_{\mathbb{C}} B_1^{\otimes N}$, which gives a time evolution as

$$T: \ B_1^{\otimes N} \to B_1^{\otimes N}$$
$$T: |c(t)\rangle \mapsto |c(t+1)\rangle.$$

At a glance, one may think that $|c(t+1)\rangle$ becomes a linear combination of many of the tensor products of B_1 crystals. However, one tensor product of B_1 crystals maps to a unique tensor product of B_1 and the resultant state exactly corresponds to the state of the pBBS at time step $t+1$. Even for the pBBS with M ($M \geqslant 2$) kinds of balls and various box capacities, the above lattice model is also well defined as far as the dimension of the vertical crystal is large enough, that is, $\kappa_t \gg 1$ for the vertical crystal B_{κ_t} of $U'_q(A^{(1)}_{M-1})$. We will show a proof of this fact for the case with one kind of balls. Since the evolution rule for M kinds of balls is decomposed into M steps as far as $\kappa_t \gg 1$, and only one kind of balls are moved at each step according to the same evolution rule, the proof is also true for the case with many kinds of balls. When κ is small, however, the above construction will not give a unique tensor product and will not define an evolution rule of the pBBS.

Since we treat only one kind of balls, the states are represented by a $U'_q(A^{(1)}_1)$ crystal. First we consider the isomorphism $B_\kappa \otimes B_\theta \simeq B_\theta \otimes B_\kappa$ given by the combinatorial R-matrix. A state b in B_κ is usually denoted by a single raw semistandard Young tableaux of length κ on letters 1 and 2. Instead we denote $b = (y, \kappa - y)$ where y is the number of 1 in the Young tableaux. For $(y, \kappa - y) \otimes (x, \theta - x) \simeq (x', \theta - x') \otimes (y', \kappa - y')$, we have the relation [12]

$$x' = y - \min[\kappa, x + y] + \min[\theta, x + y] \tag{19}$$
$$y' = x + \min[\kappa, x + y] - \min[\theta, x + y]. \tag{20}$$

For $\kappa > \theta$, the relation is explicitly written as

$$y' = \begin{cases} x & (x + y \leqslant \theta) \\ 2x + y - \theta & (\theta < x + y \leqslant \kappa) \\ x + \kappa - \theta & (\kappa < x + y) \end{cases} \tag{21}$$
$$x' = x + y - y'. \tag{22}$$

Now let θ_n ($n = 1, 2, \ldots, N$) be the capacity of the nth box, and κ_t be the capacity of the carrying cart at time step t. The state at time step t is given by $|c(t)\rangle \in B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}$. Since B_{θ_n} is an $U'_q(A^{(1)}_1)$ crystal, a vector $b_n \in B_{\theta_n}$ is represented as $b_n = (x_n, \theta_n - x_n)$, where x_n corresponds to the number of the balls in the nth box. We denote a state $b_1 \otimes b_2 \otimes \cdots \otimes b_N$ by $[x_1, x_2, \ldots, x_N]$ for $b_i = (x_i, \theta_i - x_i)$ ($i = 1, 2, \ldots, N$). The combinatorial R-matrix of $U'_q(A^{(1)}_1)$ gives the isomorphism $\mathcal{T}$:

$$\mathcal{T}: B_{\kappa_t} \otimes (B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}) \simeq (B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}) \otimes B_{\kappa_t}$$
$$([y_0] \otimes [x_1, x_2, \ldots, x_N] \simeq [x'_1, x'_2, \ldots, x'_N] \otimes [y'_0]).$$

From equation (22), we have the following recurrence equations:

$$
\begin{aligned}
y_n &= F(y_{n-1}; x_n, \theta_n) \\
&:= \begin{cases} x_n & (x_n + y_{n-1} \leqslant \theta_n) \\ 2x_n + y_{n-1} - \theta_n & (\theta_n < x_n + y_{n-1} \leqslant \kappa_t) \\ x_n + \kappa_t - \theta_n & (\kappa_t < x_n + y_{n-1}) \end{cases} \\
x'_n &= \begin{cases} y_{n-1} & (x_n + y_{n-1} \leqslant \theta_n) \\ \theta_n - x_n & (\theta_n < x_n + y_{n-1} \leqslant \kappa_t) \\ y_n - \kappa_n + \theta_n & (\kappa_t < x_n + y_{n-1}) \end{cases} \\
& \qquad\qquad\qquad\qquad (n = 1, 2, \ldots, N) \\
y'_0 &= y_N.
\end{aligned} \tag{23}
$$

We see that the function $F(y; x, \theta)$ is a piecewise linear and monotonically increasing function of y which satisfies $F(y+1; x, \theta) - F(y; x, \theta) = 0$ or 1 and $0 \leqslant F(y; x, \theta) \leqslant \kappa_t$. Since y'_0 is a function of y_0, we denote it by

$$
y'_0 = F_N(y_0; \{x_i\}, \{\theta_i\}) := \underbrace{(F \circ F \circ \cdots \circ F)}_{N \text{ times}}(y_0).
$$

The function F_N is also a monotonically increasing piecewise linear function, and $F_N(y_0 + 1; \{x_i\}, \{\theta_i\}) - F_N(y_0; \{x_i\}, \{\theta_i\}) = 0$ or 1 and $0 \leqslant F_N \leqslant \kappa_t$. Thus, for $0 \leqslant y_0 \leqslant \kappa_t$, there is one and only one integer y^* or one and only one finite interval $[y_*, y^*]$ $(y_*, y^* \in \mathbb{Z})$ where the identity $y_0 = F_N(y_0; \{x_i\}, \{\theta_i\})$ holds. Furthermore, from equations (23), we find $\{x'_n\}$ do not vary for $y_* \leqslant y_0 \leqslant y^*$. Therefore we conclude that, for given $\{x_i\}$ and $\{\theta_i\}$, there is at least one y_0 $(0 \leqslant y_0 \leqslant \kappa_t)$ at which y'_0 is equal to y_0 and that $\{x'_i\}$ are uniquely determined and have the same values for y_0 which satisfies $y'_0 = y_0$. The above conclusion means that $T := \mathrm{Tr}_{B_{\kappa_t}} \mathcal{T} \in \mathrm{End} B_{\theta_1} \otimes \cdots \otimes B_{\theta_N}$ maps a state $b_1 \otimes \cdots \otimes b_N$ to the state which is also described by one tensor product of crystal basis. We summarize the above statement as a theorem.

Theorem 2. *If κ is greater than any θ_i $(i = 1, 2, \ldots, N)$, the map $T := \mathrm{Tr}_{B_\kappa} \mathcal{T} \in \mathrm{End} B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}$ sends a tensor product of crystal basis to a unique tensor product of crystal basis of $U'_q(A_1^{(1)})$. Furthermore, for sufficiently large κ, the statement holds for $U'_q(A_M^{(1)})$ with arbitrary positive integer M.*

From the construction of the map, it is clear that the pBBS discussed in the previous section corresponds to the case $\theta_i = 1$ $(i = 1, 2, \ldots, N)$, and we use this map to construct the pBBS with arbitrary box capacities and ball species.

4.2. pBBS as $A_{N-1}^{(1)}$ crystal chains

When there are M balls of the same kind and N boxes, we can also reformulate the pBBS in terms of the combinatorial R-matrix of $U'_q(A_{N-1}^{(1)})$ and symmetric tensor product B_M and $B_{M'}$ where $M' := \sum_{i=1}^N \theta_i - M$ (figure 5).

A crystal $b \in B_M$ can be denoted by $b = (x_1, x_2, \ldots, x_N)$ with $0 \leqslant x_i \leqslant \theta_i, \sum_{i=1}^N x_i = M$. We associate a state of pBBS with the crystal b in which x_i is the number of balls in the ith box of the state. For the crystal b, we define the dual crystal $\bar{b} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N) \in B_{M'}$, where $\bar{x}_i = \theta_i - x_i$ $(i = 1, 2, \ldots, N)$. Then the crystal $b' \in B_M$ associated with the state at time $t+1$

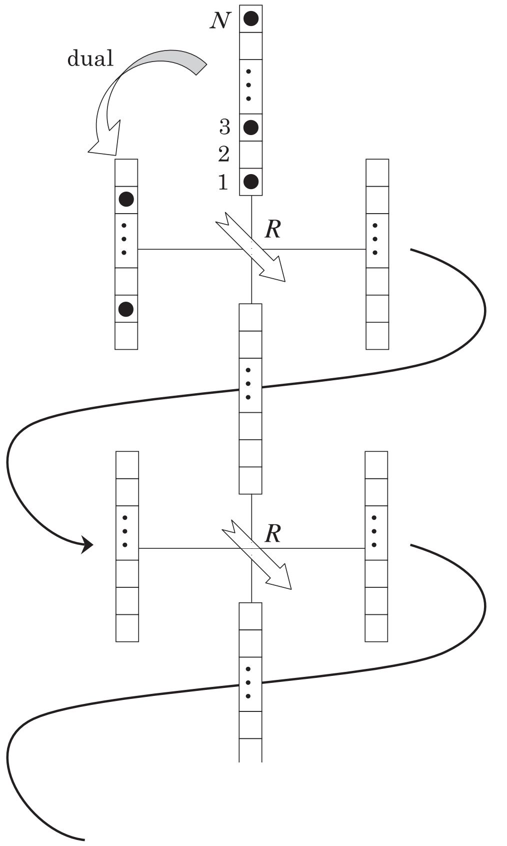


Figure 5. Twisted chains of crystal $A^{(1)}_{N-1}$ and pBBS.

is given by the combinatorial R matrix which gives the isomorphism $B_{M'} \otimes B_M \simeq B_M \otimes B_{M'}$ as

$$R: \; \bar{b} \otimes b \rightarrow b' \otimes \bar{b}'. \tag{24}$$

From [12], we see that this gives the same time evolution of the pBBS discussed above. As is shown in figure (5), the time evolution is described by two twisted chains of B_M and $B_{M'}$. Note that, by changing the crystal b and/or the $\bar{b}$ with another crystal (say a B type crystal), we obtain other types of pBBS with a time evolution rule given by the isomorphism R. We may find interesting features in these CAs. However, the investigation of these pBBSs is a problem for the future.

The isomorphism (24) has been shown to be expressed as an ultradiscrete KP equation (equations (22) and (23) in [13]), which gives another reason why we claim the pBBS is an integrable CA. Here we do not repeat the results from [13], but show a similar formulae to (2). The space of the states, however, is no longer a finite field but $\mathbb{Z}^N$. For $X = (x_1, x_2, \ldots, x_N),\ Y = (y_1, y_2, \ldots, y_N) \in \mathbb{Z}^N$, we define max and min: $\mathbb{Z}^N \times \mathbb{Z}^N \rightarrow \mathbb{Z}^N$ as

$$(\min[X, Y])_i = \min[x_i, y_i]$$
$$(\max[X, Y])_i = \max[x_i, y_i].$$

We also define the rotate shift to the right S: $\mathbb{Z}^N \to \mathbb{Z}^N$ as

$$SX = (x_N, x_1, x_2, \ldots, x_{N-1}).$$

Let $\theta_i (\in \mathbb{Z}_{>0})$ be the capacity of the ith box and $x_i(t)$ $(0 \leqslant x_i(t) \leqslant \theta_i)$ be the number of balls in the ith box at time step t. We denote the state of the pBBS at t by $X(t) := (x_1(t), x_2(t), \ldots, x_N(t))$. The state at $t+1$, $X(t+1)$, is obtained from the following theorem:

Theorem 3. *Let $A^{(0)} = X(t)$ and $B^{(0)} = SX(t)$. We define $A^{(n)}$ and $B^{(n)}$ $(n = 1, 2, \ldots)$ by the recurrence equations:*

$$\begin{aligned} A^{(n+1)} &:= \min[A^{(n)} + B^{(n)}, \boldsymbol{\theta}] \\ B^{(n+1)} &:= S\max[A^{(n)} + B^{(n)} - \boldsymbol{\theta}, \mathbf{0}], \end{aligned} \tag{25}$$

where $\boldsymbol{\theta} := (\theta_1, \theta_2, \ldots, \theta_N)$. Then we obtain

$$X(t+1) = A^{(N-1)} - X(t), \qquad B^{(N-1)} = \mathbf{0}. \tag{26}$$

Proof 3. The proof of theorem 3 is similar to that of theorem 1 and we simply show its outline. We define $D^{(i)}$ $(i = 1, 2, \ldots, N)$ as in the proof of theorem 1. Then we have the decomposition:

$$X(t) = \sum_{i=1}^{N-1} D^{(i)}, \qquad X(t+1) = \sum_{i=1}^{N-1} S^i D^{(i)}. \tag{27}$$

From the recurrence equations (25) and the properties of the similar diagram of evolution patterns, we can inductively show

$$\begin{aligned} A^{(k)} &= A^{(k-1)} + S^k D^{(k)} \\ S^{-k-1} B^{(k)} &= S^{-k} B^{(k-1)} - D^{(k)} \\ S^{k+1} D^{(k+1)} &= \min[\boldsymbol{\theta} - A^{(k)}, B^{(k)}], \end{aligned} \tag{28}$$

and we have

$$A^{(n)} = X(t) + \sum_{i=1}^{n} S^i D^{(i)}, \tag{29}$$

$$B^{(n)} = S^{n+1}\left(\sum_{i=n+1}^{N-1} D^{(i)}\right), \tag{30}$$

which completes the proof. □

5. Summary

In this paper, we introduced the pBBS which is an extension of the original BBS to that with a periodic boundary condition. We showed that the evolution rule of the pBBS is given by a recurrence Boolean formula which is regarded as an ultradiscretized algorithm of the calculation of the $2N$th root of a given number. We also gave the conserved quantities of the recurrence formula. The relation to the combinatorial R-matrix of $U'_q(A_{N-1}^{(1)})$ was clarified and the generalization of the pBBS with the symmetric tensor product representaion of $U'_q(A_{N-1}^{(1)})$ crystals was discussed.

Since the pBBS takes only a finite number of states, it has a fundamental cycle. Determination of this fundamental cycle is one of the problems for future work. In addition, an integrable equation usually has quasi-periodic solutions given by theta functions. A state of the pBBS is expected to be obtained from the quasi-periodic solution through UD. To obtain a rigorous expression of the solutions to the pBBS through UD is another future problem.

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